PMM U.S.S.R.,Vol.47,No.1,pp.134-135,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

UDC 531.381

AN EXCEPTIONAL CASE OF MOTION OF THE KOVALEVSKAIA GYROSCOPE*

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A case of degeneracy of the Kovalevskaia solution /1/ is studied. It is shown that when the integration constants of the first two integrals are both zero and that of the reduced Kovalevskaia integral is equal to four, a unique pendulum motion of a specified type occurs, and no other motions being possible.

1. Initial equations. We write the Euler - Poisson equations for the Kovalevskaia case in the form

$$\frac{dP}{d\tau}=\frac{1}{2}QR,\quad \frac{dQ}{d\tau}=-\frac{1}{2}RP-\gamma'',\quad \frac{dR}{d\tau}=2\gamma' \tag{1.1}$$

$$\frac{d\gamma}{d\tau}=R\gamma'-Q\gamma'',\quad \frac{d\gamma'}{d\tau}=P\gamma''-R\gamma,\quad \frac{d\gamma''}{d\tau}=Q\gamma-P\gamma'$$

$$\left(\tau=\nu t,\ P=\frac{p}{\nu},\quad Q=\frac{q}{\nu},\quad R=\frac{r}{\nu},\quad \nu=\sqrt{\frac{Mg\cdot OG}{A}}\right)$$

Equations (1.1) admit four first algebraic integrals

$$2(P^2+Q^2)+R^2-4\gamma=4H,\ 2(P\gamma+Q\gamma')+R\gamma''=2L \tag{1.2}$$

$$\gamma^2+\gamma'^2+\gamma''^2=1,\ (P^2-Q^2+2\gamma)^2+4(PQ+\gamma')^2=K^2$$

We assume that the integration constants of the kinetic energy and kinetic moment of the body about the vertical (directed, for definiteness, downward) are all equal to zero, and the integration constant of the fourth integral is equal to four

$$H=L=0,\quad K^2=4 \tag{1.3}$$

According to the classification of G.G. Appel'rot the case in question belongs to the "particularly noticeable motion of fourth class" /3/ even though Appel'rot himself did not study this case.

2. Set S of initial conditions: $P_0=0$. The equations (1.2) and conditions (1.3) mean that the following four conditions are imposed on the initial values $(\tau=0)$ of the variables:

$$2(P_0^2+Q_0^2)+R_0^2=4\gamma_0,\ 2(P_0\gamma_0+Q_0\gamma_0')+R_0\gamma_0''=0 \tag{2.1}$$

$$\gamma_0^2+\gamma_0'^2+\gamma_0''^2=1,\quad (P_0^2-Q_0^2+2\gamma_0)^2+4(P_0Q_0+\gamma_0')^2=4$$

The conditions define the set S of initial conditions in the space $P_0, Q_0, R_0, \gamma_0, \gamma_0', \gamma_0''$. Incidentally, since equations (2.1) hold for any $\tau>0$, the motion must remain within the set S investigated in the phase space $P, Q, R, \gamma, \gamma', \gamma''$. From the first equation of (2.1) we have

$$\gamma_0\geqslant 0,\ 2(P_0^2+Q_0^2)+R_0^2\leqslant 4$$

and in particular

$$R_0^2\leqslant 4,\ P_0^2+Q_0^2\leqslant 2,\ |P_0|\leqslant\sqrt{2},\ |Q_0|\leqslant\sqrt{2}$$

Let us put in (2.1) $P_0=0$ and substitute the expression for γ_0 and $\gamma_0'^2$, obtained from the first and last equation, into the third equation of (2.1). This yields

$${}^1/_4Q_0^2(Q_0^2+R_0^2)+\gamma_0''^2=0$$

and consequently we find that $Q_0=\gamma_0''=0$. Substituting these last equalities into (2.1) we obtain

$$P_0=0:Q_0=\gamma_0''=0,\quad R_0^2=4\gamma_0,\quad \gamma_0^2+\gamma_0'^2=1\quad (|R_0|\leqslant\sqrt{2}) \tag{2.2}$$

where all subsequent equations follow from the first equation. Thus the segment (2.2) of the hypercurve of one dimension from which the following pendulum motion appears, belongs to the set S :

*Prikl.Matem.Mekhan.,Vol.47,No.1,pp.166-168,1983

$$p = Q = \gamma'' \equiv 0, \quad R^2 \equiv 4\gamma, \quad \gamma^2 + \gamma'^2 \equiv 1 \quad (|R| \leqslant \sqrt{2})$$

The motion is executed about the extended axis of the ellipsoid of revolution occupying a horizontal position. The center of mass G of the body is raised in this dimension to the horizontal plane passing through the fixed point. We note that neither position of equilibrium belongs to the set S and the steady rotations as well as the remaining pendulum type motions are not generated from it.

3. Set $S: P_0 \neq 0$. Assume that $R_0 = 0$. Then, obtaining the expressions for γ_0 and γ_0' from the first and second equation of (2.1) (noting that $Q_0 \neq 0$ since $P_0\gamma_0 \neq 0$), we obtain

$$2\gamma_0 = P_0^2 + Q_0^2, \quad \gamma_0' = -\frac{P_0}{Q_0}\gamma_0$$

Substituting these expressions into the third and fourth equation of (2.1) we find that

$$Q_0^3/P_0^2 + \gamma_0''^2 = 0$$

which is impossible, therefore

$$R_0 \neq 0 \quad \text{when} \quad P_0 \neq 0 \tag{3.1}$$

Next we shall show that the system of equations (2.1) has no solutions when $P_0 \neq 0$. Writing the expression for γ_0 from the first equation of (2.1) and substituting it into the fourth equation of (2.1), we obtain

$$\gamma_0' = -P_0Q_0 \pm \sqrt{1 - (P_0^2 + {}^1/_4R_0^2)^2} \quad (P_0^2 + {}^1/_4R_0^2 \leqslant 1) \tag{3.2}$$

where the sign preceding the radical is governed by the initial value of γ_0'. Then, from the third equation of (2.1) we obtain

$$\gamma_0''^2 = {}^3/_4P_0^4 - {}^3/_2P_0^2Q_0^2 - {}^1/_4Q_0^4 + {}^1/_4(P_0^2 - Q_0^2)R_0^2 \pm 2P_0Q_0\sqrt{1 - (P_0^2 + {}^1/_4R_0^2)^2} \tag{3.3}$$

Next we write the second equation of (2.1) in the form

$$2(P_0\gamma_0 + Q_0\gamma_0') = -R_0\gamma_0''$$

and square it. Substituting the expressions for γ_0, γ_0' and γ_0'' from the first equation of (2.1), (3.2), (3.3), we now obtain a biquadratic equation for R_0 which in turn yields

$$R_0^2 = -\frac{4}{(P_0^2 + Q_0^2)^4}[2Q_0(P_0^2 - Q_0^2) \pm P_0\sqrt{(P_0^2 + Q_0^2)^4 - 16Q_0^4}]^2$$

The expression within the square brackets will be purely imaginary only for conditions $Q_0 = P_0$, $P_0^2 < 1$, in which case we have

$$R_0^2 = 4\frac{1 - P_0^4}{P_0^2}, \quad P_0^2 + \frac{1}{4}R_0^2 = P_0^{-2} > 1$$

which, by virtue of (3.2), does not correspond to the real motion. Thus we have shown that in the case of $P_0 \neq 0$ no motions of the Kovalevskaia gyroscope are possible. It follows therefore that the set S of initial conditions for the equations of motion (1.1) of the Kovalevskaia gyroscope consists, in the exceptional case defined by the conditions (2.1), only of (2.4), and apart from (2.3) no other motions are possible.

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Translated by L.K.